

THE SYMMETRY AXIOM IN MINKOWSKI PLANES

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Dedicated to H. Karzel

ABSTRACT. The aim of the paper is to give a synthetic proof that in a symmetric Minkowski plane the Benz's (G) axiom holds (without using the algebraic representation).

INTRODUCTION

The foundations of Minkowski planes were built in the late sixties by R. Artzy, W. Benz, W. Heise, H. Karzel and others (cf. [1], [3], [4], [7], [8]). Two large and well-investigated classes of models of Minkowski planes are determined by the symmetry axiom (S) (cf. [7]) and the rectangle axiom (G) ([3]).

It is well known that the axiom (S) determines the class of miquelian planes over commutative fields and the axiom (G) the larger class of planes connected with so called Tits near-fields ([6]). It follows that the symmetry axiom implies the rectangle axiom but all known proofs have been indirect. They involve the algebraic representation of the planes.

As a generalization of Minkowski planes by omitting the touching axiom (T) hyperbola structures are considered (cf. [3], [7]). The geometry of the graphs of a sharply 3-transitive permutation set is a hyperbola structure (cf. [7]). The rectangle axiom is equivalent to the fact that the permutation set describing the hyperbola structure is closed under composition (cf. [3]). Hyperbola structures fulfilling (S) correspond to symmetric sharply 3-transitive permutation sets.

In the fine paper [13] Karzel proved that every symmetric sharply 3-transitive permutation set is isomorphic to $\text{PGL}(2, K)$, where K is a commutative field, hence the corresponding hyperbola structure is a symmetric Minkowski plane and satisfies the rectangle axiom (G).

The synthetic proof that the touching axiom (T) is a consequence of the symmetry axiom is given in [7].

The aim of this paper is to present a direct, synthetic proof of the implication $(S) \rightarrow (G)$ and to explain the geometric connection between the two axioms. Our proof is long, but this point of view provides a natural and intrinsic characterization of orthogonality in symmetric Minkowski planes. In some considerations we use ideas of Karzel from [13] and give them a geometric interpretations.

In Section 2 we give a synthetic proof that (S) implies both conditions of Dienst ([5])

(\star) Every symmetry with respect to any circle is an automorphism.

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(**) The composition of three symmetries with respect to circles of a bundle is the symmetry with respect to a circle (of this bundle).

In Section 3 we define Minkowski planes of characteristic two independently of the notion of the characteristic of translation planes. For other Minkowski planes we introduce harmonic relation of generators. Then we define and characterize an involutory automorphism with two pointwise fixed generators called harmonic homology.

In Section 4 we obtain some configuration theorems which can be regarded as special cases of the rectangle axiom with orthogonality of suitable circles in assumptions. Especially the configurations from Lemma 4.1 and Corollary 4.4 appeared useful for a description of ordered Minkowski planes in our next prepared paper. In this section we use the language of permutation set for a brief exposition of results. However all presented statements have clear geometric interpretations.

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1. NOTATIONS AND BASIC DEFINITIONS.

Let \mathcal{P} be a non empty set of *points* and $\Lambda, \Sigma_1, \Sigma_2$ non empty subsets of the power set of \mathcal{P} with $\Sigma_1 \cap \Sigma_2 = \emptyset$; the elements of Λ and $\Sigma_1 \cup \Sigma_2$ will be called *circles* and *generators* respectively. $\mathcal{M} := (\mathcal{P}, \Lambda, \Sigma_1 \cup \Sigma_2)$ is called *hyperbola structure* if the following conditions are valid:

- (H1) For each point $p \in \mathcal{P}$ and $i \in \{1, 2\}$ there is exactly one generator $G \in \Sigma_i$ with $p \in G$ (notation $[p]_i := G$).
- (H2) Any two generators A, B with $A \in \Sigma_1, B \in \Sigma_2$ intersect in exactly one point.
- (H3) Every generator intersects every circle in exactly one point.
- (H4) For any three distinct points $a, b, c \in \mathcal{P}$ with $[a]_i \neq [b]_i \neq [c]_i \neq [a]_i$ for $i \in \{1, 2\}$ there is exactly one circle $C \in \Lambda$ such that $a, b, c \in C$.
- (H5) There exists a circle containing at least three points.

For any point $p \in \mathcal{P}$ the incidence structure $\mathcal{M}^p := (\mathcal{P}^p, \Lambda^p)$, where $\mathcal{P}^p := \mathcal{P} \setminus ([p]_1 \cup [p]_2)$ and $\Lambda^p := \{K \setminus \{p\} \mid K \in \Lambda, p \in K\} \cup \{A \setminus [p]_2 \mid A \in \Sigma_1, p \notin A\} \cup \{B \setminus [p]_1 \mid B \in \Sigma_2, p \notin B\}$ is called *the derived structure of \mathcal{M} in the point p* (cf. [7]).

A hyperbola structure $\mathcal{M} = (\mathcal{P}, \Lambda, \Sigma_1 \cup \Sigma_2)$ is called a *Minkowski plane* if it has the property:

- (T) (*Touching axiom*) For any circle $C \in \Lambda$, any $a \in C$ and any $b \in \mathcal{P} \setminus (C \cup [a]_1 \cup [a]_2)$ there is exactly one circle $B \in \Lambda$ with $b \in B$ and $C \cap B = \{a\}$.

\mathcal{M} is a Minkowski plane if and only if the derived structure \mathcal{M}^p in any point p is an affine plane.

For points p, q and $K \in \Lambda$ we write $[p] = [p]_1 \cup [p]_2$, $pq = [p]_1 \cap [q]_2$, $pK = K \cap [p]_1$, $Kp = K \cap [p]_2$. For $p, q, r \in \mathcal{P}$ such that $p \notin [q]$, $q \notin [r]$, $p \notin [r]$ the unique circle through p, q, r is denoted by $(p, q, r)^\circ$. A set of points X is called *concyelic* if there exists a circle $K \in \Lambda$ such that $X \subset K$. If $p \notin [q]$, the set $\langle p, q \rangle := \{K \in \Lambda \mid p, q \in K\}$ is called the *bundle* with vertices p, q . If $p \in K$, for some circle K , the set $(p, K) = \{L \in \Lambda \mid p \in L, L \cap K = \{p\}\}$ is called the *pencil* determined by the vertex p and the circle K (the pencil of tangent circles with vertex p and direction K). Two points p, q are called *symmetric with respect to a circle K* if $pq, qp \in K$. The point symmetric to p with respect to a circle K is denoted by $S_K(p)$. For a circle K the involutory bijection S_K fixing the points

of K and exchanging the sets Σ_1, Σ_2 is called the *circle symmetry with respect to K* . Two different circles K, L are called *orthogonal* if $S_K(p) = p$ for every point $p \in L$ (notation: $K \perp L$).

In this paper we consider only hyperbola structures satisfying the following symmetry axiom (cf. [3], [4], [7]):

(S) For any two circles K, L and a point $p \in L \setminus K$ if $S_K(p) \in L$, then $K \perp L$.

If a hyperbola structure satisfies (S), then it is a Minkowski plane (cf. [7]), so we call it the *symmetric Minkowski plane*. Until further notice we assume that $\mathcal{M} := (\mathcal{P}, \Lambda, \Sigma_1 \cup \Sigma_2)$ is a symmetric Minkowski plane. We repeat this assumption only in the theorems.

The main idea of the paper is to present a synthetic proof that every symmetric Minkowski plane fulfils the following rectangle axiom (cf. [3], [4])

(G) Let $\{p_i | 1 \leq i \leq 4\}, \{q_i | 1 \leq i \leq 4\}$ be two concyclic sets of different points such that $\{p_i q_i | 1 \leq i \leq 4\}$ is also a concyclic set of points. Then the set of points $\{q_i p_i | i = 1, \dots, 4\}$ is concyclic.

Since the proof of the implication (S) \rightarrow (T) given in [7] is synthetic, next we will use the touching axiom.

2. PROPERTIES OF SYMMETRIES WITH RESPECT TO A CIRCLE

We use the following connection between orthogonality and tangency in the pencil of tangent circles with vertex p .

Proposition 2.1. *Let K, L, M be different circles such that $p \in K, L, M$, $K \perp M$. Then $L \perp K \Leftrightarrow L \cap M = \{p\}$.*

Proof. \Rightarrow See [7], Satz 6.

\Leftarrow Let $x \in L$, $x \neq p$ and $x' = S_K(x)$. The circle $L' = (x, p, x')^\circ$ is orthogonal to K and tangent to M at p by (\Rightarrow) . We obtain $L' = L$ by the touching axiom. \square

In order to make all the constructions possible we assume that every circle contains at least 8 points. This is no loss of generality because in another case the plane is easy to describe (cf. [9]).

Proposition 2.2. *If $I, J \in \Lambda$ and $I \cap J = \{p\}$, then $S_I(J)$ is a circle.*

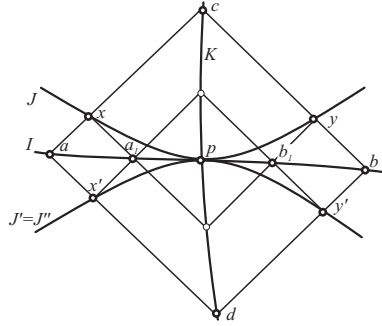


FIGURE 1.

Proof. (fig. 1) Let $x \in J \setminus p$, $x' = S_I(x)$, and let J' be the unique circle passing through x' and tangent to J at p . We show that $y' = S_I(y) \in J'$ for any $y \in J$.

Let $a = xI$, $b = yI$. Suppose first that $a \neq b$.

Let $c = ab$, $d = ba$ and $K = (c, p, d)^\circ$. By (S), $K \perp I$. Since $I \cap J = \{p\}$, we have $K \perp J$ by Proposition 2.1 and the points x, y are symmetric with respect to K . If $a_1 = Ix$ and $b_1 = yI$, then $Ka_1 = Kx = yK = b_1K$ and so a_1, b_1 are symmetric with respect to K . Since $x'K = a_1K = Kb_1 = Ky'$ and $Kx' = Ka = bK = y'K$, it follows that x', y' are symmetric with respect to K . Consider the circle $J'' = (x', p, y')^\circ$. By Proposition 2.1, $J'' \cap I = \{p\}$ because $J'' \perp K$. According to the touching axiom, $J'' = J'$ and $y' \in J'$.

If $a = b$, it is enough to consider the point $z \in J$ such that $Iz \neq a$ and $zI \neq yI$. By the first part of the proof we conclude that $S_I(z) \in J'$ and finally that $y' \in J'$. \square

Proposition 2.3. *If $I, J \in \Lambda$ and $I \cap J = \{a, b\}$ ($a \neq b$), then $S_I(J)$ is a circle.*

Proof. (fig. 2) For any point $x \in J$, $x \neq a, b$, set $x' = S_I(x)$. Let $J' = (a, b, x')^\circ$ and y be an arbitrary point such that $y \in J$, $y \neq a, b, x$. Then $y' = S_I(y) \in J'$.

Indeed, let $c = ab$, $d = ba$, $e = xy$, $f = yx$ and $K = (c, d, e)^\circ$. By (S), we obtain $K \perp I, J, J'$ and $f \in K$. We define $e_1 = eI$, $e_2 = Ie$, $f_1 = If$, $f_2 = fI$. As $I \perp K$, we deduce that e_1, e_2 and f_1, f_2 are symmetric with respect to K . So we have $Ky' = Kf_2 = f_1K = x'K$ and $y'K = e_2K = Ke_1 = Kx'$. Thus y' is symmetric to x' with respect to K and by (S), $y' \in J'$. \square

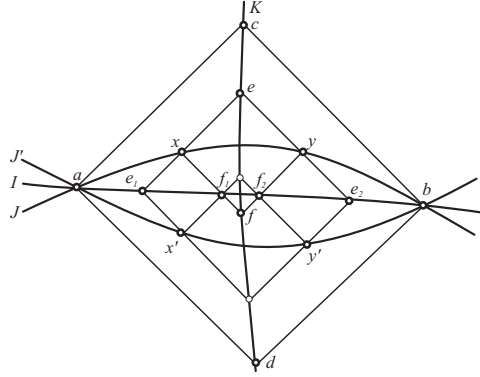


FIGURE 2.

Lemma 2.1. *Let $I, J \in \Lambda$ and $x, y \in J \setminus I$ be different points. If $x_1 = J(xI)$ and $x_2 = J(x_1I)$, then the points $S_I(x)$, $S_I(y)$, $S_I(x_1)$, $S_I(x_2)$ are concyclic.*

Proof. (fig. 3) We put $a = Ix$, $b = yI$, $c = ab$, $d = ba$, $e = S_J(d)$ and $K = (c, d, e)^\circ$. By (S), $K \perp I$. We set $b_1 = Iy$, $f = b_1a$, $g = ab_1$, $h = S_J(f)$ and $L = (f, g, h)^\circ$. By construction, it follows from (S) that $K, L \perp I, J$. We introduce the following points: $y_1 = b_1J$, $b_2 = Iy_1$, $y_2 = b_2J$, $b_3 = Iy_2$, $a_1 = xI$, $a_2 = x_1I$, $a_3 = x_2I$. Then $y_1 = S_L(x)$. The points a_1 and b_1 are symmetric with respect to K because $a_1K = Kb_1 = e$ and $I \perp K$. Analogously, we show successively that the pairs of points (x_1, y_1) , (a_2, b_2) , (x_2, y_2) , (a_3, b_3) are symmetric with respect to K . In the same way we show that (x, y_1) , (a_1, b_2) , (x_1, y_2) , (a_2, b_3) are symmetric with respect to L . Let $x' = S_I(x)$, $y' = S_I(y)$, $x'_i = S_I(x_i)$, $y'_i = S_I(y_i)$ ($i = 1, 2$). Then the pairs

(x', y') , (x'_i, y'_i) ($i = 1, 2$) are symmetric with respect to K and the pairs (x', y'_1) , (x'_1, y'_2) are symmetric with respect to L . Indeed, for example for the pair (x'_1, y'_2) we have the equalities

$$x'_1 L = a_1 L = L b_2 = L y'_2,$$

$$L x'_1 = L a_2 = b_3 L = y'_2 L,$$

because the pairs (a_1, b_2) , (a_2, b_3) are symmetric with respect to L . From these equalities we see that x'_1 and y'_2 are symmetric with respect to L .

Consider the circle $M = (x', y', y'_1)^\circ$. Then $M \perp K, L$ because $S_K(x') = y'$ and $S_L(x') = y'_1$. Using the previous symmetries we get consecutively $x'_1 \in M$ (because $x'_1 = S_K(y'_1)$), $y'_2 \in M$ and $x'_2 \in M$. \square

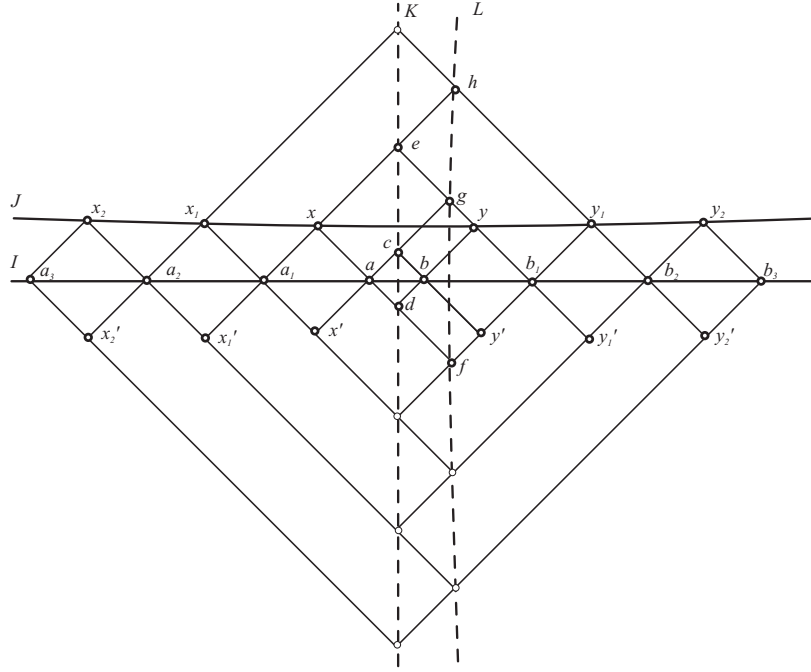


FIGURE 3.

Theorem 2.1. *For a symmetric Minkowski plane the symmetry with respect to an arbitrary circle is an automorphism.*

Proof. Let $J \in \Lambda$. If $I \cap J \neq \emptyset$, then $S_I(J)$ is a circle by Propositions 2.2 and 2.3. Assume that $I \cap J = \emptyset$. Let $x \in J$ and x_1, x_2 be as in Lemma 2.1. Then Lemma 2.1 shows that $S_I(y) \in (S_I(x), S_I(x_1), S_I(x_2))^\circ$ for all $y \in J$. \square

Corollary 2.1. *Every symmetry with respect to a circle preserves orthogonality of circles.*

Theorem 2.2. *Let \mathcal{M} be a symmetric Minkowski plane. If $I, J, K \in \langle p, q \rangle$ for some points p, q , then there exists $L \in \langle p, q \rangle$ such that $S_L = S_K \circ S_J \circ S_I$.*

Proof. (fig. 4) Let $r = pq$, $s = qp$. For every $x \notin [p] \cup [q]$ consider the points $x_1 = S_I(x)$, $x_2 = S_J(x_1)$, $x_3 = S_K(x_2)$ and the circle $M = (r, s, x)^\circ$. According to

(S), $x_1, x_2, x_3 \in M$. If $a = xx_3$, $b = x_3x$, then a and b are symmetric with respect to M . For $L = (p, q, a)^\circ$, $b \in L$ because $L \perp M$. We show that for any point y , $S_L(y) = S_K \circ S_J \circ S_I(y)$.

For the points p, q, r, s, x this follows immediately from the construction. Assume that $y \notin [p] \cup [q] \cup [x]$. Let $y_1 = S_I(y)$, $y_2 = S_J(y_1)$, $y_3 = S_K(y_2)$ and $N = (r, s, y)^\circ$. Then $y_1, y_2, y_3 \in N$. If $c = yy_3$, $d = y_3y$, $z = yx = cb$, then $P = (r, s, z)^\circ \perp I, J, K$. It follows that the points $z_1 = S_I(z)$, $z_2 = S_J(z_1)$, $z_3 = S_K(z_2)$ belong to P . We obtain successively the chain of equalities:

$z_1z = x_1x$, $zz_1 = yy_1$, $z_1z_2 = x_1x_2$, $z_2z_1 = y_2y_1$, $z_3z_2 = x_3x_2$, $z_2z_3 = y_2y_3$. From the first and fifth equality we get $z_3z = b$, and from the second and sixth $zz_3 = c$, so $S_P(b) = c$. Analogously, $S_P(a) = d$. Since $P \perp L$, it follows that $c, d \in L$ and $y_3 = S_L(y)$.

If $y \in [x]$, the above arguments apply with some simplifications ($z = x$ or $z = y$).

If $y \in [p] \cup [q]$, then without loss of generality we assume that $y \in [p]_1, y \neq p, r$. For any $a \in [y]_2 \setminus [p]$ by the previous part of the proof we get $S_K S_J S_I(a) = S_L(a)$. It follows that $S_K S_J S_I([y]_2) = S_L([y]_2)$ because $[a]_2 = [y]_2$. We also have $S_K S_J S_I([y]_1) = S_L([y]_1)$ because $[y]_1 = [p]_1$, so $S_K S_J S_I(y) = S_L(y)$. \square

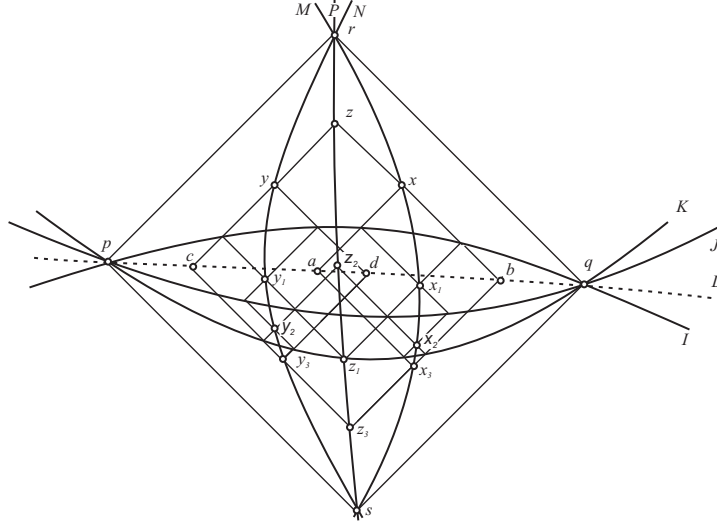


FIGURE 4.

Just as for bundles, we have the theorem on reduction for pencils of tangent circles.

Proposition 2.4. *If $I, J \in (p, K)$, then there exists $L \in (p, K)$ such that $S_L = S_K \circ S_J \circ S_I$.*

Proof. The composition $S_K \circ S_J \circ S_I$ is an automorphism which induces on the derived affine plane \mathcal{M}^p a central collineation with improper center corresponding to the ideal point of the pencil of circles orthogonal to K in p . The axis of the collineation is a proper line (because each symmetry exchanges the sets Σ_1 and Σ_2), so we get a pointwise fixed circle passing through p , tangent to K . \square

3. CHARACTERISTIC OF A MINKOWSKI PLANE AND HARMONIC RELATION OF GENERATORS OF THE SAME KIND

Proposition 3.1. *If $I, J \in \Lambda$, then $I \perp J \Leftrightarrow S_J \circ S_I$ is an involution.*

Proof. (fig. 5) \Rightarrow Since $I \perp J$, for $x \in I \cup J$, we have $S_J \circ S_I(x) = S_I \circ S_J(x)$. If $x \notin I \cup J$, we set $x' = S_I(x)$, $x'' = S_J(x')$ and $x_1 = S_I(x'')$. We show that $x = S_J(x_1)$.

Let $a = xI$, $b = Ix$, $c = x''I$, $d = Ix''$. We obtain $Ja = Jx' = x''J = cJ$, so the points c and a are symmetric with respect to J . Similarly, b and d are symmetric with respect to J . We obtain $xJ = aJ = Jc = Jx_1$ and $Jx = Jb = dJ = x_1J$. Hence x, x_1 are symmetric with respect to J .

\Leftarrow For any $x \in I$ we have $S_I \circ S_J(x) = S_J \circ S_I(x) = S_J(x)$, hence $S_J(x) \in I$. \square

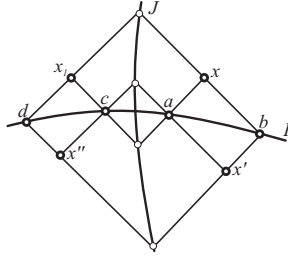


FIGURE 5.

Definition 3.1. ([11], Def. 2.2) An automorphism φ of a Minkowski plane is called a *double homothety* if there exist points p, q , ($p \notin [q]$) such that $p, q, pq, qp \in \text{fix } \varphi$ and $\varphi|_{\mathcal{M}^p}$, $\varphi|_{\mathcal{M}^{pq}}$ are homotheties of affine planes.

Corollary 3.1. ([11], Prop. 2.2, p. 422) *If $I, J \in \Lambda$, $I \perp J$, $I \cap J = \{a, b\}$, then $S_J \circ S_I$ is a double homothety.*

Proposition 3.2. *If $I, J, K \in \Lambda$, $I \perp J, K$, $I \cap J = \{p\}$ and $p \in K$, then $I \cap K = \{p\}$.*

Proof. Since $I \cap J = \{p\}$ and $I \perp K$, we have $K \perp J$ by Proposition 2.1. As $K, I \perp J$ and $p \in I, J, K$, it follows that $K \cap I = \{p\}$ by Proposition 2.1. \square

Proposition 3.3. *If $I \cap J = \{p, q\}$ and $I \perp J$, then for every circle $K \in \langle p, q \rangle$ there exists exactly one $L \in \langle p, q \rangle$ such that $L \perp K$.*

Proof. By Theorem 2.2, there exists $L \in \langle p, q \rangle$ such that $S_L = S_K \circ S_J \circ S_I$. So we get $S_J \circ S_I = S_K \circ S_L$ and $K \perp L$ by Proposition 3.1. If $L, L' \in \langle p, q \rangle$, $L, L' \perp K$ and $L' \neq L$, we obtain a contradiction with Proposition 2.1. \square

The following theorem allows us to introduce the definition of symmetric Minkowski planes of characteristic 2.

Theorem 3.1. *Let \mathcal{M} be a symmetric Minkowski plane. If there exists a pair of orthogonal and tangent circles, then every pair of tangent circles is orthogonal and every pair of orthogonal circles has at most one common point.*

Proof. First we prove the second part of the theorem. Since any pair of different points can be transformed in any other by a composition of two symmetries and each symmetry preserves orthogonality (Corollary 2.1), it is enough to show that the following is impossible: $I \perp J$, $I \cap J = \{p\}$ and $K \perp L$, $K, L \in \langle p, q \rangle$ for some $I, J, K, L \in \Lambda$, $p, q \in \mathcal{P}$, $p \notin [q]$, $q \in I$.

Let K' be a circle such that $q \in K'$, $K' \cap J = \{p\}$. By Proposition 3.3, there exists a circle $L' \in \langle p, q \rangle$ with $L' \perp K'$. Then $L' \perp I, J$ and by Proposition 2.1, L' is tangent to K' , contrary to $L' \in \langle p, q \rangle$.

To prove the first part assume that $I \cap J = \{p\}$. If $K \in \Lambda$, $p \in K$, $K \neq I$ and $K \perp J$, then by the already proven part of the theorem we get $K \cap J = \{p\}$. In particular $K \cap I = \{p\}$ and $K \perp J$. From Proposition 2.1 we obtain $I \perp J$. \square

By Theorem 3.1, we can introduce the following definition.

Definition 3.2. A symmetric Minkowski plane \mathcal{M} is of characteristic 2 (notation $\text{char}\mathcal{M} = 2$) if there exists a pair of orthogonal tangent circles.

We use the notation $\text{char}\mathcal{M} \neq 2$ in the case \mathcal{M} is not of characteristic 2 although we don't define another characteristic of Minkowski planes. This doesn't arise confusion and agrees with the usually used notation (cf. [5], [6], [7]).

From the definitions we remark that $\text{char}\mathcal{M} \neq 2$ exactly when for some points p, q with $p \notin [q]$ there exists a double homothety with centers p, q (cf. [11]). For such planes the existence of one homothety implies that all the possible double homotheties exist.

From Proposition 3.1 and [11, Proposition 2.2], we get the following characterization of involutory homotheties of symmetric Minkowski planes.

Proposition 3.4. *If φ is an automorphism of a Minkowski plane \mathcal{M} with $\text{char}\mathcal{M} \neq 2$, then the following are equivalent:*

- (i) φ is an involutory homothety,
- (ii) φ is a double homothety,
- (iii) φ is a superposition of two symmetries with respect of two orthogonal intersecting circles.

Remark 3.1. The equivalence of (i) and (ii) is proved in [12] without a further assumption.

By Proposition 3.4 and [12, Corollary 2] if $\text{char}\mathcal{M} \neq 2$ for any a, b ($a \notin [b]$) there exists exactly one double homothety with centers a, b which will be denoted by $\mathbf{H}_{a,b}$.

Lemma 3.1. *If $\text{char}\mathcal{M} \neq 2$ and points a, b, c, d satisfy the conditions $[a]_i = [c]_i$, $[b]_i = [d]_i$ for a fixed i ($i = 1, 2$), then $\mathbf{H}_{a,b}(X) = \mathbf{H}_{c,d}(X)$ for every $X \in \Sigma_i$.*

Proof. First we assume $a = c$. The automorphism $\varphi = \mathbf{H}_{a,d} \circ \mathbf{H}_{a,b}$, being a superposition of two homotheties, induces a collineation with improper axis of the derived affine plane \mathcal{M}^a . If $\varphi' = \varphi|_{\mathcal{M}^a}$ had a proper center, then there would exist $e \in [a]_i$, $f \in [b]_i \cap [e]_{3-i}$ such that $\varphi(e) = e$, $\varphi(f) = f$. Then φ induces a homothety with center f on the plane \mathcal{M}^a . Analogously φ induces a homothety with center e of the plane $\mathcal{M}^{a'}$, where $\{a'\} = [a]_{3-i} \cap [b]_i$.

Hence, $\varphi = \mathbf{H}_{a,f}$ would be an involution and $\mathbf{H}_{a,d}(b)$ a fixed point of $\mathbf{H}_{a,b}$, a contradiction. It follows that the center of φ' is an improper point corresponding to

the class Σ_1 and $\mathbf{H}_{a,b}(X) = \mathbf{H}_{a,d}(X)$ for every $X \in \Sigma_i$. If $a \neq c$, then by the above, for every $X \in \Sigma_i$ we get $\mathbf{H}_{a,b}(X) = \mathbf{H}_{c,b}(X) = \mathbf{H}_{c,d}(X)$. \square

For $\mathcal{M} = (\mathcal{P}, \Lambda, \Sigma_1 \cup \Sigma_2)$ with $\text{char} \mathcal{M} \neq 2$, $A, B \in \Sigma_i$ and $A \neq B$ we may put the following two definitions.

Definition 3.3. $X, Y \in \Sigma_i$ are *harmonic conjugate* with respect to A, B if there exist $a \in A$, $b \in B$ such that $\mathbf{H}_{a,b}(X) = Y$ and $X, Y \neq A, B$.

Definition 3.4. The *harmonic homology* with axis A, B (denoted by $\mathbf{S}_{A,B}$) is a permutation of \mathcal{P} defined as follows:

- (i) for $x \in A \cup B$, $\mathbf{S}_{A,B}(x) = x$
- (ii) for $x \notin A \cup B$, $\mathbf{S}_{A,B}(x) = y \Leftrightarrow [x]_{3-i} = [y]_{3-i}$ and the generators $[x]_i, [y]_i$ are harmonic conjugate with respect to A, B .

By Lemma 3.1 Definition 3.3 is independent of the points a, b .

Corollary 3.2. For any $A, B \in \Sigma_i$ such that $A \neq B$, $\mathbf{S}_{A,B}$ is an involutory automorphism of the Minkowski plane.

Proof. $\mathbf{S}_{A,B}$ is an involutory bijection preserving the sets Σ_i because double homothety is an involution. Let $K \in \Lambda$, $a = A \cap K$, $b = B \cap K$. We show that $\mathbf{S}_{A,B}(K) = L$, where $L \in \Lambda$, $a, b \in L$ and $L \perp K$. By Proposition 3.4, $\mathbf{H}_{a,b} = S_L \circ S_K$. If $x \in K$, $x' \in L$ and $[x]_{3-i} = [x']_{3-i}$, then $\mathbf{S}_{A,B}([x]_i) = S_L \circ S_K([x]_i) = S_L([x]_{3-i}) = S_L([x']_{3-i}) = [x']_i$ and $x' = \mathbf{S}_{A,B}(x)$. \square

Corollary 3.3. If $\text{char} \mathcal{M} \neq 2$, $X, Y \in \Sigma_i$, $X \neq Y$, $K, L \in \Lambda$, then $\mathbf{S}_{X,Y}(K) = L \Leftrightarrow L \perp K \wedge K \cap L = K \cap (X \cup Y)$.

We now turn to the case of characteristic two.

Proposition 3.5. Let $\text{char} \mathcal{M} = 2$. If $X \neq Y$, $X, Y \in \Sigma_i$ and $p \notin X \cup Y$, then there exists an involutory translation exchanging X, Y and fixing pointwise $[p]_i$.

Proof. Let $K \in \Lambda$ with $p \in K$ and x, x' be different points of K with $x, x' \neq p$. The circle $L = (p, xx', x'x)^\circ$ is orthogonal to K . By Definition 3.2 and Theorem 3.1 L is tangent to K in a point p . The automorphism $S_L \circ S_K$, according to Proposition 3.1, is an involution exchanging x, x' , and induces a translation of \mathcal{M}^p . In the same manner we can obtain the transitivity of translations of \mathcal{M}^p in any other direction determined by a circle.

By [10, Theorem 4.19, p. 100], we obtain the transitivity of translations with the direction determined by Σ_{3-i} . From the proof of [10, Theorem 4.19] it follows that these translations are compositions of translations with directions determined by circles, so they are induced by automorphisms of \mathcal{M} . They are also involutions by [10, Theorem 4.14, p. 97]. \square

Remark 3.2. From Corollary 3.2 and Proposition 3.5 it follows that on a symmetric Minkowski plane for different generators $A, X, Y \in \Sigma_i$ there exists an involutory automorphism which fixes A pointwise and exchanges X and Y (see [6] for this fact for Minkowski planes with axiom (G)).

Proposition 3.6. Let $\text{char} \mathcal{M} = 2$. If circles K, L are tangent at p and $M \perp K, L$, then $M \in (p, K)$.

Proof. If $p \notin M$, we obtain a contradiction with Theorem 3.1 because the points $p, S_M(p)$ are two different common points of the orthogonal circles K, L . \square

4. THE PROOF OF POSTULATE G

We begin with the main lemma.

Lemma 4.1. *Let $p, q \in \mathcal{P}$ and $K, L, M, N \in \Lambda$ satisfy the following conditions:*

(i) $p \neq q$, $p \in [q]_2$, $p \in K \cap L$, $q \in M \cap N$,

(ii) $K, M \perp L, N$.

Then the conditions $k \in K$, $l \in L$, $m \in M$, $n \in N$ and $k \in [l]_1$, $l \in [m]_2$, $m \in [n]_1$ imply $k \in [n]_2$.

Proof. (fig. 6) Assume that $\text{char} \mathcal{M} \neq 2$. We define the points $r = pM$, $s = pN$, $t = qL$ and $u = qK$. By orthogonalities of suitable circles, we get $r \in [t]_2$ and $s \in [u]_2$. For $x \in K \cap L$, $x \neq p$ consider the harmonic homology $\varphi = \mathbf{S}_{P,X}$, where $P = [p]_2$, $X = [x]_2$. By definition of φ , we get $\varphi(K) = L$ and $\varphi(t) = u$. Hence $\varphi(s) = r$ and $\varphi(M) = N$. By assumptions, $\varphi(l) = k$, $\varphi(m) = n$ and $l \in [m]_2$. We finally get $k \in [n]_2$.

In the case $\text{char} \mathcal{M} = 2$ the proof is analogous. Instead of the automorphism $\mathbf{S}_{P,X}$ we use a translation with pointwise fixed generator P which maps t to u ; such a translation exists by Proposition 3.5 and is involutory. \square

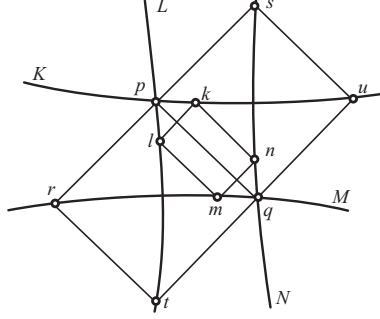


FIGURE 6.

Lemma 4.2. *Let p, r, s, q, a, b, c, d be different points and K, L, M, N be different circles such that $s = pr$, $q = rp$, $b = ca$, $d = ac$, $K, M \in \langle p, r \rangle$, $L, N \in \langle q, s \rangle$, $a \in L$, $b \in M$, $c \in N$, $d \in K$.*

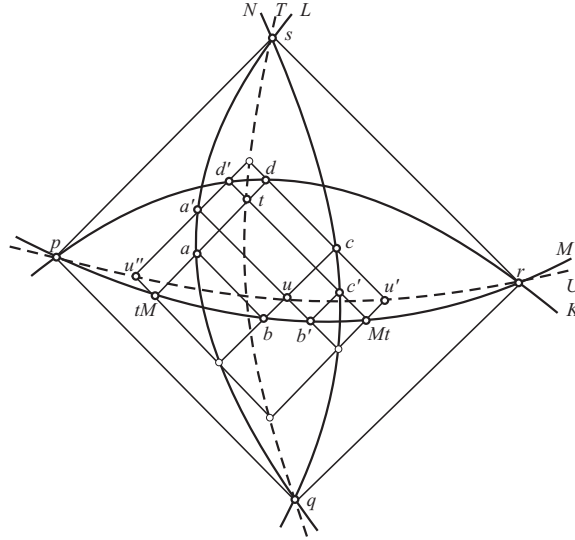
If $a' \in L$, $b' \in M$, $c' \in N$ and $b' = c'a'$, $d' = a'c'$, then $d' \in K$.

Proof. (fig. 7) Consider the points $t = dd'$, $u = bb'$ and the orthogonal circles $T = (s, t, q)^\circ$, $U = (u, p, r)^\circ$. If $u' = \mathbf{S}_N(u)$, $u'' = \mathbf{S}_L(u)$, then $u', u'' \in U$ because $L, N \perp U$. Hence $\mathbf{S}_T(u') = u''$ and $\mathbf{S}_T(d) = d'$. It follows that $d' \in K$ because $T \perp K$. \square

In the remainder of this section we fix an arbitrary circle E .

It is well known that every $K \in \Lambda$ determines the permutation $f_K : E \rightarrow E$ by the formula $f_K(x) = (Kx)E$ for $x \in E$ and vice versa (cf. [3], [6], [7], [13]). By definition, $f_E = \text{id}_E$. According to (S), if $K \perp E$, then $f_K^2 = \text{id}_E$. The involution corresponding to a circle K , $K \perp E$ will be denoted by i_K . According to Theorem 2.1, for any f_K , there exists the permutation $f_K^{-1} := f_{K'}$, where $K' = S_E(K)$.

Now we are going to show that the permutation set $\{f_K | K \in \Lambda\}$ is closed under composition. This is equivalent to the rectangle axiom.



Corollary 4.1. (i) *If $L, N \in \Lambda$, $q \in E \cap (N \setminus L)$ and $L, N \perp E$, then there exists $K \in \Lambda$ such that $Lq \in K$, $K \perp L, N$ and $f_K = i_L \circ i_N$.*
(ii) *If $\text{char} \mathcal{M} \neq 2$ or $K \not\perp E$, then for every $q \in E \setminus K$ there exist circles L, N such that $f_K = i_L \circ i_N$, $q \in N$, $Kq \in L$ and $L, N \perp K$.*

(i) If $L \not\perp N$, then we set $K := (qL, S_N(qL), S_L(S_N(qL)))^\circ$. If $L \perp N$ and $\text{char} \mathcal{M} \neq 2$ the unique circle K such that $K \in \langle qL, S_N(qL) \rangle$ and $K \perp L$ (Proposition 3.3) is the circle we are looking for. The case $\text{char} \mathcal{M} = 2$ and $L \perp N$ is impossible by Proposition 3.6.

Remark 4.1. Corollary 4.1 enables one to represent any permutation f_K as a composition of two involutions one of which has a fixed point (in the case $\text{char } \mathcal{M} = 2$ with the assumption that f_K is not an involution).

Analogously to Corollary 4.1, from Lemma 4.2, we obtain

Corollary 4.2. (i) If $L, N \in \langle q, s \rangle$, $L, N \perp E$ and $E \notin \langle q, s \rangle$, then there exists $K \in \Lambda$ such that $K \in \langle qE, Eq \rangle$ and $f_K = i_L \circ i_N$.
(ii) If $K, E \in \langle p, r \rangle$ and $c \notin [p] \cup [r]$, then there exist circles L, N such that $c \in N$, $L, N \in \langle pr, rp \rangle$ and $f_K = i_L \circ i_N$.

Proposition 4.1. *If $L, N \perp E$ and $p \in E \cap L \cap N$, then there exists a circle K such that $K \cap E = \{p\}$ and $i_N \circ i_L = f_K$.*

Proof. Let $\text{char}\mathcal{M} \neq 2$. There exists a point $q \neq p$ such that $q \in E \cap L$ and $q \notin N$ by Proposition 2.1 (fig. 8). From part (i) of Corollary 4.1 it follows that there exists $K \in \Lambda$, $K \perp L, N$ such that $i_L \circ i_N = f_K$. Since $p \in L \cap N$, we have $f_K(p) = p$ and $K \cap E = \{p\}$.

If $\text{char}\mathcal{M} = 2$, by Lemma 2.1 there exists $K \in \Lambda$ such that $S_K = S_L \circ S_E \circ S_N$. If $x \in E$ and $y = Nx$, then $S_K(y) = S_L S_E(y)$ (fig. 9). Hence $(L(yE))K = Ky$ and this means that $i_L \circ i_N(x) = f_K(x)$. We remark that in this case f_K is an involution. \square

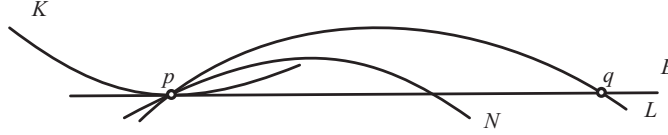


FIGURE 8.

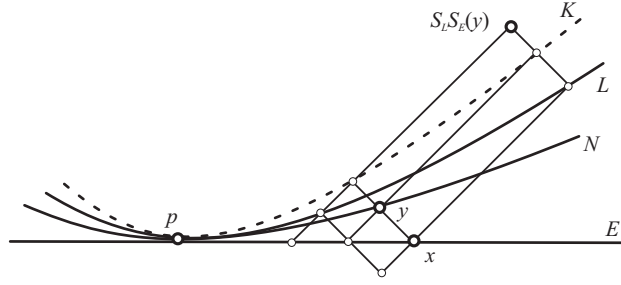


FIGURE 9.

Corollary 4.3. *If $E \cap K = \{p\}$, $L \perp E$, $p \in L$, then $i_L \circ f_K = i_N$ for some $N \in \Lambda$ such that $p \in N$.*

Proof. If $\text{char}\mathcal{M} = 2$, this is obvious. Let $x \in E$, $y = Kx$, $q = (L(yE))y$ and let N be a circle tangent to L at p and passing through q (fig. 10). By Proposition 4.1, there exists $K' \in \Lambda$ such that $i_L \circ i_N = f_{K'}$ and $K' \cap E = \{p\}$. From the construction of $f_{K'}$ we get $y \in K'$, so $K = K'$. \square

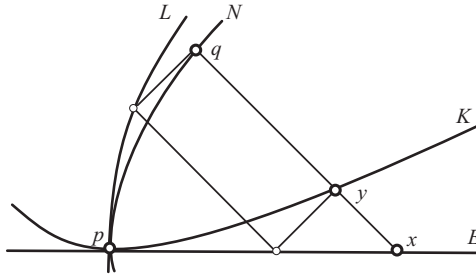


FIGURE 10.

Corollary 4.4. *If $p \in E \cap K \cap L \cap M$ and $K, L, M \perp E$, then $i_M \circ i_L \circ i_K = i_N$ for some $N \in \Lambda$ through p .*

Now we will collect some propositions concerning superpositions of the bijections f_K . We have not been able to find a shorter way to prove the postulate (G). The main problem is the case of nonintersecting circles.

Proposition 4.2. *If $L \perp E$ and $p \in L \cap E$, then for every circle K there exists a circle M such that $f_K \circ i_L = f_M$.*

Proof. Assume that $p \notin K$. From Corollary 4.1(ii), $f_K = i_S \circ i_R$, where $R, S \perp E$ and $p \in R$, $Kp \in S$ (fig. 11). If $\text{char}\mathcal{M} = 2$, then there exists $T \in \Lambda$ such that $p \in T$ and $i_R \circ i_L = i_T$ by Proposition 4.1. Hence $f_K \circ i_L = i_S \circ i_R \circ i_L = i_S \circ i_T$. According to Corollary 4.1(i), there exists a circle M such that $i_S \circ i_T = f_M$. If $\text{char}\mathcal{M} \neq 2$, then by Corollary 4.1(ii), there exist $T, U \in \Lambda$ such that $i_S = i_U \circ i_T$, where $p \in T$. By Corollary 4.4, we find a circle W such that $i_W = i_T \circ i_R \circ i_L$. Hence $f_K \circ i_L = i_S \circ i_R \circ i_L = i_U \circ i_T \circ i_R \circ i_L = i_U \circ i_W$. By Corollary 4.1(i), there exists a circle M such that $f_K \circ i_L = i_U \circ i_W = f_M$.

Let now $p \in K$ and $\text{char}\mathcal{M} \neq 2$. Then we can find $q \neq p$ such that $q \in L \cap E$. If $q \in K$, the result follows from Corollary 4.2. If $q \notin K$, it follows from the already proven part of the Proposition.

Assume $p \in K$ and $\text{char}\mathcal{M} = 2$. If $K \cap L = \{p\}$, then f_K is an involution and the result follows from Corollary 4.3. In the other case let us consider the circle $K' := S_E(K)$. We have $K' \cap L = \{p, q\}$ with $q \notin E$ and we set: $a = Eq$; r such that $r \in K' \cap E$, $r \neq p$; $m = pr$; $n = rp$ (fig. 12). If $b \neq a, p, r$ is an arbitrary point of E , then $f_{K'} = i_G \circ i_F$, where $F = (m, n, b)^\circ$, $G = (m, n, K'b)^\circ$ by Corollary 4.1(ii). If Q is a circle such that $f_Q = i_G \circ i_L$ (Corollary 4.1), then $b \notin Q$. Indeed, if $b \in Q$, then $S_G(b) \in Q \cap E$ (because $G, L \perp Q, E$), hence $E = (S_G(b), S_L(b), b)^\circ = Q$ and we get a contradiction because $n \in Q$ and $n \notin E$. It follows that $b \notin S_E(Q) = Q'$. So we have $i_L \circ f_{K'} = i_L \circ i_G \circ i_F$, hence $i_L \circ f_{K'} = f_Q^{-1} \circ i_F = f_{Q'} \circ i_F = f_{M'}$ for some circle M' by the first part of the proof. For the circle $M := S_E(M')$, we obtain $f_K \circ i_L = f_M$. \square

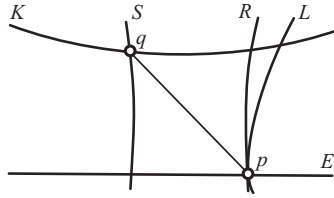


FIGURE 11.

Proposition 4.3. *If $L \perp E$ and $p \in L \cap K$, then there exists a circle M such that $f_M = i_L \circ f_K$.*

Proof. By Corollary 4.1(ii), there exist $R, S \perp E, K$ such that $f_K = i_S \circ i_R$, $p \in S$ and $Ep \in R$. Then $i_L \circ f_K = i_L \circ i_S \circ i_R$. As $i_L \circ i_S = f_T$ for some T by Corollary 4.2, we conclude that $i_L \circ f_K = f_T \circ i_R = f_M$ for some $M \in \Lambda$ by Proposition 4.2. \square

- [6] Hartmann, E. Minkowski-Ebenen über scharf 3-fach transitiven Permutationsgruppen. Habilitationsschrift, Darmstadt, 1979.
- [7] Heise, W., Karzel, H. Symmetrische Minkowski-Ebenen. J. Geometry 3 (1973), 5-20.
- [8] Heise, W., Karzel, H. Vollkommen Fanosche Minkowski-Ebenen. J. Geometry 3 (1973), 21-29.
- [9] Heise, W., Seybold, H. Das Existenzproblem der Möbius-, Laguerre- und Minkowski-Erweiterungen endlicher affiner Ebenen. Bayer. Akadem. Wiss. Math.-Natur. Kl. S.-B. 1975 43-58.
- [10] Hughes, D., Piper, F. Projective Planes. Springer-Verlag, 1970.
- [11] Jakóbowski, J., Matraś, A. Multicentral automorphisms in geometries of circles. Bull. Polish Acad. of Sci. 49, (2001), 417-432.
- [12] Jakóbowski, J., Kroll H.-J., Matraś, A. Minkowski planes admitting automorphism groups of small type. J. Geom. 71 (2001), 78 - 84.
- [13] Karzel, H. Symmetrische Permutationsmengen. Aequationes Mathematicae, 17 (1978), 83-90.

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